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AN IMPLICIT NUMERICAL SCHEME FOR FRACTIONAL ADVECTION DIFFUSION EQUATION

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ABSTRACT

In this paper, a finite difference scheme is presented for time fractional advection diffusion equation (TFADE). This equation is derived from classical advection diffusion equation with variable coefficients on replacing classical integer order derivatives by their fractional counterpart. An advection diffusion equation describes physical phenomenon where particle, energy or other physical quantities are transferred inside a physical system due to combined effect of advection and diffusion. To address anomalous diffusion like sub diffusion or super diffusion, classical integer derivatives are replaced by corresponding fractional derivative to obtain TFADE. Using central difference approximations for both space and time fractional derivatives, it is found the present numerical scheme is unconditionally stable.

Keywords: *Finite Difference Method, Advection-Diffusion Equation, Fractional Derivatives.*

I. INTRODUCTION

Fractional Calculus has find its applications in various field of sciences and engineering viz. fluid dynamics, hydrology, acoustic field etc. along with commercial fields such as finance and marketing. Differential equations of arbitrary order (i.e. fractional order) had been used widely to model various systems, material processes, physical and chemical phenomenon. Apart from integer order derivatives, fractional order derivatives address various anomalous phenomenon like sub diffusion and super diffusion. Many attempts are made to find out analytical as well as numerical solution of fractional partial differential equations (FPDE) governing these phenomenon. Among these, sub diffusion and super diffusion processes occurring due to fractional derivative, are studied by many authors (e.g. [2], [3], [10], [12], [14], [15]). Langland and Henry [8] developed numerical algorithm for time fractional diffusion equation of the form $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}=\frac{\partial^\alpha u(x,t)}{\partial x^2}$. Considering Riemann Liouville fractional derivative, Yuste [13] proposed weighted average finite difference method for force free diffusion equation represented by $\frac{\partial u(x,t)}{\partial t}=K_0 D_t^{1-\gamma}\frac{\partial^\alpha u(x,t)}{\partial x^2}$. For $0<\alpha<1$, Murio [11] developed unconditionally stable fully implicit scheme for time fractional diffusion equation of the form $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}=\frac{\partial^\alpha u(x,t)}{\partial x^2}$ in the space domain $0<x<1$ Ding and Li [5] presented a class of weighted finite difference methods for numerical solution of space fractional diffusion equation. Spectral methods based numerical scheme was presented by Lin and Xu [9]. Liu and Yang [10] proposed a numerical method for anomalous sub diffusion equation having a non linear source term. Zhang et.al [14] presented an unconditionally stable finite difference scheme for space time fractional convection diffusion equation of the form $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}=a(x).\frac{\partial^\beta u(x,t)}{\partial t^\beta}-b(x)=\frac{\partial u(x,t)}{\partial x}-c(x)u(x,t)+q(x,t)$ with with initial and boundary conditions given by u(0,t)=u(L,t)=0. Meerschaert et.al. [12] presented finite difference scheme for fractional equation on two time scales. Zhang et.al [15] carried out estimation of error by using fully discrete scheme based on reduction of order. By transforming partial differential equations into an equivalent partial integro differential equation, Huang et.al. [6] constructed two finite difference schemes for time fractional diffusion wave equations. Zhi et.al [4] discussed numerical scheme for one dimensional fractional advection dispersion equation of the form $\frac{\partial C}{\partial t}+U\frac{\partial C}{\partial x}=K\frac{\partial^2 C}{\partial x^2}$. Yang et.al. [6] proposed a meshless implicit method for 2D time dependent fractional diffusion wave equation. Using fractional shifted Legendre polynomial method, hybrid numerical scheme was proposed by Krishnveni et.al [7]. By using an approximation to Reisz fractional derivative, Arshad et.al.[1] presented a finite difference scheme to obtain the numerical solution of time space fractional advection diffusion equation of the form ${}_{C}D_t^{\alpha}\,u(x,t)=K_{\beta_1}\frac{\partial^{\beta_1}}{\partial|x|^{\beta_1}}u(x,t)+K_{\beta_2}\frac{\partial^{\beta_2}}{\partial|x|^{\beta_2}}u(x,t)+f(x,t)$. The advection diffusion equation is generally used to model many physical, chemical biological, oceanographic and meteorological processes. The general form of the advection-diffusion equation is $\frac{\partial u}{\partial t}+v\frac{\partial u}{\partial x}=\frac{\partial^2 u}{\partial x^2}$, where $u(x,t)$ represents scalar fields like temperature or concentration, $v(x,t)$ represents adjective fluid velocity and K is diffusion constant. In the present paper, unconditionally stable fully implicit numerical scheme is presented for time fractional advection diffusion equation (TFADE) of the form

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}+a\frac{\partial u(x,t)}{\partial x}=\frac{\partial}{\partial x}\left(v(x,t)\frac{\partial}{\partial x}u(x,t)\right) \qquad (1)$$

On simplification, (1) gets converted into

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}=\frac{\partial v(x,t)}{\partial x}\frac{\partial u(x,t)}{\partial x}+v(x,t)\frac{\partial^2 u(x,t)}{\partial x}-a\frac{\partial u(x,t)}{\partial x} \qquad (2)$$

To the best of our knowledge no numerical scheme is proposed for this form of advection diffusion equation. In the present scheme, space domain is restricted to [0,1]. Along with initial conditions, Dirichlet Boundary conditions are used.

II. NUMERICAL SOLUTION OF TFADE

This section deals with the numerical approximation of TFADE using Caputo fractional time derivative order $0<\alpha<1$ given by

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}=\frac{1}{\Gamma(n-\alpha)}\int_0^t\frac{\partial u(x,t)}{\partial t}(t-z)^{-\alpha}\,dz, \qquad 0\le t\le T,\quad 0<\alpha<1$$

Let h and k be the mesh size in space and time respectively such that h=1/M and k=1/N, where M,N represent number of subintervals in space and time partition respectively. With these implications spatial grid points are given by $x_{j}=jh$ where j=0,1,...,M and grid points in time interval [0,1]are given by $t_r=rk$ where $r=0,1,\ldots TN$. Numerical approximation of the solution of TFADE at each grid point is given by $u_j^r=u(x_j,t_r)$ whereas the value of function f at (x_j,t_r) is denoted by $f_j=f(x_j)$. Now we discretize (2) on grid points (x_j,t_r), we have

$$\frac{\partial^{\alpha} u(x_j,t_r)}{\partial t^{\alpha}} = \frac{\partial v(x_j,t_r)}{\partial x}\frac{\partial u(x_j,t_r)}{\partial x} + v(x_j,t_r)\frac{\partial^2 u(x_j,t_r)}{\partial x} - a\frac{\partial u(x_j,t_r)}{\partial x} \quad (3)$$

Using numerical approximation to time fractional derivative of first order as suggested by Murio [11] we have

$$\frac{\partial^{\alpha} u(x_j,t_r)}{\partial t^{\alpha}} = D_t^{\alpha} u_j^r + O(k) \quad (4)$$

Where

$$D_t^{\alpha} u_i^n = \sigma_{\alpha,k}\sum_{r=1}^{r=n} w_r^{\alpha}\,(u_i^{n-r+1} - u_i^{n-r}) \quad and \quad \sigma_{\alpha,k} = \frac{1}{(1-\alpha)\Gamma(1-\alpha)}\frac{1}{k^{\alpha}} \quad (5)$$

Using central difference approximations to $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial^2 u}{\partial x^2}$, we have

$$\frac{\partial v(x_j,t_r)}{\partial x} = \frac{v(x_{j+1},t_r) - v(x_{j-1},t_r)}{2h} + O(h^2) \quad (6)$$

$$\frac{\partial u(x_j,t_r)}{\partial x} = \frac{u(x_{j+1},t_r) - u(x_{j-1},t_r)}{2h} + O(h^2) \quad (7)$$

$$\frac{\partial^2 u(x_j,t_r)}{\partial x^2} = \frac{u(x_{j-1},t_r) - 2u(x_j,t_r) + u(x_{j+1},t_r)}{h^2} + O(h^2) \quad (8)$$

If U_j^r and V_j^r denote the numerical approximations of $u(x_j,t_r)$ and $v(x_j,t_r)$ respectively, then substituting equations (6), (7) and (8), equation (3) reduces to

$$\sigma_{\alpha,k}\sum_{r=1}^{r=n} w_r^{(\alpha)}(U_j^{n-r+1} - U_j^{n-r} = \left[\frac{V_{j+1}^r - V_{j-1}^r}{2h}\right].\left[\frac{U_{j+1}^r - U_{j-1}^r}{2h}\right] + V_j^r\left[\frac{U_{j-1}^r - 2U_J^R + U_{j+1}^r}{h^2}\right]$$
$$- a\left[\frac{U_{j+1}^r - U_{j-1}^r}{2h}\right]$$
$$= \left[\frac{V_{j+1}^r - V_{j-1}^r}{4h^2} + V_j^r - \frac{a}{2h}\right]U_{j+1}^r + \left[\frac{V_{j-1}^r - V_{j+1}^r}{4h^2} + V_j^r - \frac{a}{2h}\right]U_{j-1}^r - 2V_j^r U_j^r$$

With boundary conditions $U_0^n = U_M^n = 0$ and initial conditions $U_j^0 = f_j, \quad for \quad j = 1,2,\ldots M-1$

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III. STABILITY ANALYSIS OF NUMERICAL SCHEME

For stability analysis, equation is reconstituted as follows

$$\sigma_{\alpha,k}\sum_{r=1}^{r=n} w_r^{(\alpha)}(U_j^{n-r+1}-U_j^{n-r} = \left[\frac{V_{j+1}^r - V_{j-1}^r}{4h^2} + V_j^r - \frac{a}{2h}\right]U_{j+1}^r + \left[\frac{V_{j+1}^r - V_{j-1}^r}{4h^2} - V_j^r - \frac{a}{2h}\right]U_{j-1}^r - 2V_j^r U_j^r$$

$$\sigma_{\alpha,k}\sum_{r=1}^{r=n} w_r^{(\alpha)}(U_j^{n-r+1}-U_j^{n-r} = \left[\frac{V_{j+1}^r - V_{j-1}^r}{4h^2} + V_j^r - \frac{a}{2h}\right](U_{j+1}^r - U_{j-1}^r) - 2V_j^r (U_j^r - U_{j-1}^r)$$

If n=1, then r=1 and we have

$$\sigma_{\alpha,k}(U_j^1 - U_j^0 = \left[\frac{V_{j+1}^1 - V_{j-1}^1}{4h^2} + V_j^1 - \frac{a}{2h}\right](U_{j+1}^1 - U_{j-1}^1) - 2V_j^1 (U_j^1 - U_{j-1}^1)$$

For $n \geq 2$

$$\sigma_{\alpha,k}\sum_{r=1}^{r=n} w_r^{(\alpha)}(U_j^{n-r+1}-U_j^{n-r} = V_{j,r}(U_{j+1}^r - U_{j-1}^r) - 2V_j^r (U_j^r - U_{j-1}^r)$$

Where

$$V_{j,r} = \left[\frac{V_{J+1}^r - V_{j-1}^r}{4h^2} + V_j^r - \frac{a}{2h}\right]$$

$$\sigma_{\alpha,k}(U_j^r - U_j^{r-1}) + \sigma_{\alpha,k}\sum_{r=2}^{r=n} w_r^{(\alpha)}(U_j^{n-r+1}-U_j^{n-r}) = V_{j,r}(U_{j+1}^r - U_{j-1}^r) - 2V_j^r (U_j^r - U_{j-1}^r)$$

If we put $U_p^n = \xi_n e^{iwph}$, where $i = \sqrt{-1}$ and w is a real number, then equation (7) becomes

$$\sigma_{\alpha,k}(\xi_n e^{iwph} - \xi_{n-1} e^{iwph}) + \sigma_{\alpha,k}\sum_{r=2}^{r=n} w_r^{(\alpha)}(\xi_{n-r+1} e^{iwph} - \xi_{n-r} e^{iwph}) = V_{j,r}(\xi_r e^{iw(p+1)h} - \xi_r e^{iw(p-1)h}) - 2V_j^r(\xi_r e^{iwph} - \xi_r e^{iw(p-1)h})$$

Equating real part and cancelling e^{iwph} throughout we have

$$\sigma_{\alpha,k}(\xi_r - \xi_{r-1}) + \sigma_{\alpha,k}\sum_{r=2}^{r=n} w_r^{(\alpha)}(\xi_{n-r+1} - \xi_{n-r}) = -2V_j^r \xi_r (1 - \cos wh)$$

On further simplifications, the above equation can be reduced to

$$\xi_r = \frac{\sigma_{\alpha,k}\left[\xi_{r-1} + \sum_{r=2}^{r=n} w_r^{(\alpha)}(\xi_{n-r} - \xi_{n-r+1})\right]}{\sigma_{\alpha,k} + 2V_j^r(1 - \cos wh)}$$

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or

$$\xi_r = \frac{\xi_{r-1} + \sum_{r=2}^{r=n} w_r^{(\alpha)} (\xi_{n-r} - \xi_{n-r+1})}{1 + \frac{2V_j^r}{\sigma_{\alpha,k}}(1-\cos wh)}$$

Since $V(x,t) \geq 0$ and $-1 \leq \cos wh \leq 1$, $\quad 1 + \frac{2V_j^r}{\sigma_{\alpha,k}}(1-\cos wh) \geq 1$. Now equation (9) suggests that

$\xi_1 \leq \xi_0$ and for $r \geq 2$

$$\xi_r \leq \xi_{r-1} + \sum_{r=2}^{r=n} w_r^{(\alpha)} (\xi_{n-r} - \xi_{n-r+1})$$

Since for each r, $w_r^{(\alpha)} \geq 0, \quad w_r^{(\alpha)} (\xi_{n-r} - \xi_{n-r+1}) \leq w_r^{(\alpha)} \xi_{n-r}$, we have

$$\xi_r \leq \xi_{r-1} + \sum_{r=2}^{r=n} w_r^{(\alpha)} \xi_{n-r}$$

If r=2, then

$$\begin{aligned} \xi_2 &\leq \xi_1 + w_2^{(\alpha)} \xi_0 \\ &\leq \xi_0 + w_2^{(\alpha)} \xi_0 \\ &= (1 + w_2^{(\alpha)}) \xi_0 \\ &= k_2 \xi_0, \qquad where \quad k_2 = 1 + w_2^{(\alpha)} \end{aligned}$$

If r=3, then

$$\begin{aligned} \xi_3 &\leq \xi_2 + w_2^{(\alpha)} \xi_1 + w_3^{(\alpha)} \xi_0 \\ &\leq k_2 \xi_0 + w_2^{(\alpha)} \xi_0 + w_3^{(\alpha)} \xi_0 \\ &= (k_2 + w_2^{(\alpha)} + w_3^{(\alpha)}) \xi_0 \\ &= k_3 \xi_0, \qquad where \quad k_3 = k_2 + w_2^{(\alpha)} + w_3^{(\alpha)} \end{aligned}$$

Continuing like this, we have $\xi_n \leq k_n \xi_0$. Let $k = \max\{1 = k_1, k_2, k_3, \ldots k_n\}$. Then $\xi_n \leq k \xi_0$. This shows that the numerical solution is bounded by initial conditions which in turns asserts unconditional stability.

IV. CONCLUSION

The numerical solution of time fractional advection diffusion equation is computed by taking space domain [0,1] and Dirichlet boundary conditions. On the basis of theoretical computations, it is established that the proposed scheme is fully implicit and unconditionally stable.

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